Mapping graph states under local complementation[1]

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Measurement-based quantum quantum protocols—based on graph states—are at the core of modern quantum computing and communications architectures[2]. Local complementation—the graph operation that links all local-Clifford equivalent graph states—allows us to classify all stabiliser states by the entanglement they posses[3]. In this work, we study the structure of the orbits generated by local complementation, mapping them up to 9 qubits to reveal a rich hidden structure.

Graph states are quantum states with a one-to-one correspondence to mathematical graphs where the qubits are initalsed to $|+\rangle$ and for each edge (i,j) of the graph a control-Z (CZ) operation is applied between qubit i and qubit j. However, despite having obviously different constructions via nonlocal CZ operations, some graph states are locally equivalent[4,5]. Interestingly, a single graph transformation, local complementation, traverses the entire set of locally equivalent graphs. Using this simple relationship, graph state entanglement classes have been fully classified up to $n = 12$. Until now, however, the structure of the orbits generated by local complementation have not been studied. We use a depth-first search to explore every connection between locally equivalent graph states, revealing a diverse myriad of structures.

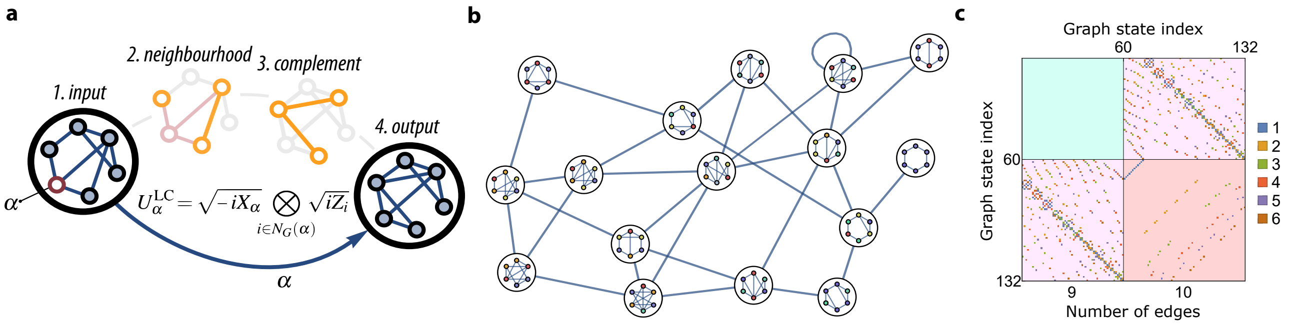

Fig. 1 local complementation and the orbits it induces. Orbit edges are labelled with the vertex that undergoes local complementation. **a.** A guide to local complementation. The neighbourhood of qubit α is complemented to yield the output graph. **b. c.** The orbit of entanglement class 18. Vertex colour denotes equivalent local complementations. **d.** The adjacency matrix of the orbit of graph state class 19.

We provide new software, freely available online, with which to compute these orbits and provide data for the first 587 orbits (up to and including 9 qubits) as well as our tools to generate them and a means to visualise them[6,7]. We study these orbits and find direct links between the connectivity of certain orbits with the entanglement properties of their component graph states. Furthermore, we observe strong correlations between graph-theoretical orbit properties, such as diameter and colourability, with entanglement monotones. We also suggest potential applications. For example, local complementation allows spatially separated qubits to be 'redistributed' in communication protocols, and can also be used to improve state preparation complexity in quantum computing architectures with a restricted two-qubit gate topology. In this way, local complementation acts to 'compile' measurement-based quantum computations.

It is well known that graph theory and quantum entanglement have a strong interplay. Our results deepen this relationship and provide a new avenue both to explore the nature of entanglement, as well as to build useful tools for quantum technology.

References

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