# Joint measurability structures realizable with qubit measurements: incompatibility via marginal surgery

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**Abstract.** Quantum measurements can be incompatible. An intuitive way to represent the (in)compatibility relations on a set of measurements is via a hypergraph representing their joint measurability structure: vertices represent measurements and hyperedges represent (all and only) subsets of compatible measurements. Projective measurements in quantum theory realize all joint measurability structures that are graphs while general measurements represented by positive operator-valued measures (POVMs) can realize arbitrary joint measurability structures. Here we explore the scope of joint measurability structures realizable with qubit POVMs. We develop marginal surgery, a technique to obtain nontrivial joint measurability structures starting from a set of compatible measurements.

**Keywords**: POVMs, Joint measurability, Joint measurability structure, Marginal Surgery

The lack of joint measurability (or compatibility) is crucial to the demonstration of nonclassicality in quantum theory, e.g., both Bell inequality violations and Kochen-Specker (KS) contextuality are impossible in the absence of incompatible measurements. While the joint measurability (or compatibility) of a set of projective measurements is a binary property, characterized entirely by their pairwise commutativity, the joint measurability of general quantum measurements given by positive operator-valued measures (POVMs) is, in general, not characterized by pairwise commutativity.

In this contribution we focus on the simplest possible nontrivial POVMs – two-outcome or binary qubit POVMs. We review the previously known conditions for their joint measurability and prove new facts and conditions for their joint measurability. We obtain these results using a technique that we term *Marginal Surgery*. It consists in the following: 1) we start from a set $s$ of compatible POVMs with some joint POVM $G^s$; 2) we obtain a joint POVM $G^r$ for some subset $r \subset s$ by coarse graining of $G^s$ over the outcomes of the POVMs in $s/r$; 3) we tweak the POVMs in $s$ to obtain some set of POVMs $s'$ of the same cardinality as $s$ such that $s'$ is incompatible and we also tweak $G^r$ into some $G^{r'}$ which is a joint POVM for $r' \subset s'$ (that corresponds to $r$). In this way we have achieved two goals: 1) we have started from a set of compatible measurements and obtained some new joint measurability structure; 2) we have derived a sufficient condition for joint measurability of the set $r'$ since the conditions for the positivity of $G^{r'}$ are the sufficient conditions for joint measurability of $r'$. Thus, marginal surgery is powerful in generating new joint measurability structures and obtaining sufficient conditions for joint measurability.

Using this technique, we obtain the following results: 1) we show that $N$-Specker scenario, a joint measurability structure on a set $N$ incompatible measurements such that each $N-1$ element subset is compatible, is realizable with binary qubit POVMs; 2) we show that all joint measurability structures with $N=4$ measurements are realizable with binary qubit measurements; 3) we present more examples of joint measurability structures realizable with binary qubit measurements with $N \in \{5,6\}$ POVMs; 4) we obtain a sufficient condition for joint measurability of arbitrary $N$ of most general binary qubit measurements (Theorem 14); 5) we obtain other sufficient conditions for joint measurability of arbitrary number of binary qubit POVMs that satisfy certain restrictions (Lemma 2, Theorem 13).

In Ref [1] it was shown that *all* joint measurability structures admit quantum realization. Critical for than construction was breaking up the structure one wants to realize into so called minimally incompatible sets of POVMs, essentially $N$-Specker scenarios. Our realization of arbitrary $N$-Specker scenarios on a qubit renders the construction in [1] maximally efficient in terms of dimension of Hilbert space required to realize a desired joint measurability structure.

Motivated by all of the various joint measurability structures we realized with binary qubit measurements we conjecture that *all* joint measurability structures admit qubit realization. This can be shown, perhaps, by some clever application of marginal surgery. We also provide a potential counterexample to refute this conjecture – a joint measurability structure that might not be qubit realizable.



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**References**

[1] R. Kunjwal, C. Heunen, and T. Fritz, Quantum realization of arbitrary joint measurability structures, Phys. Rev. A 89, 052126 (2014).



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