Approximate tensorization of the relative entropy for noncommuting conditional expectations

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[1] Probably the most fundamental property of entropy is the *strong subadditivity* inequality [7]: Given a tripartite system $\mathcal{H}_{ABC} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ and a state ρ_{ABC} on $\mathcal{H}_{ABC}$, the following holds

$$S(\rho_{ABC}) + S(\rho_B) \leq S(\rho_{AB}) + S(\rho_{BC}) \,, \tag{SSA}$$

where $S(\rho) = -\operatorname{Tr}[\rho \ln \rho]$ and for any subsystem D of ABC, $\rho_D := \operatorname{Tr}_{D^c}[\rho_{ABC}]$. Restated in terms of the relative entropy, given by $D(\rho\|\sigma) := \operatorname{Tr}[\rho\,(\ln\rho - \ln\sigma)]$, (SSA) takes the following form:

$$D\left(\rho_{ABC}\middle\|\rho_B \otimes \mathbb{1}_{AC}/d_{\mathcal{H}_{AC}}\right) \leq D\left(\rho_{ABC}\middle\|\rho_{AB} \otimes \mathbb{1}_{C}/d_{\mathcal{H}_{C}}\right) + D\left(\rho_{ABC}\middle\|\rho_{BC} \otimes \mathbb{1}_{A}/d_{\mathcal{H}_{A}}\right) . \tag{0.1}$$

In this paper [2], we derive a new generalisation of the strong subadditivity of the entropy to the setting of general conditional expectations onto arbitrary finite-dimensional von Neumann algebras. Let $\mathcal{M} \subset \mathcal{N}_1, \mathcal{N}_2 \subset \mathcal{N}$ be four von Neumann algebras and let $E^{\mathcal{M}}, E_1, E_2$ be their corresponding conditional expectations. When $E_1 \circ E_2 = E_2 \circ E_1 = E^{\mathcal{M}}$, we have:

$$D(\rho\|E^{\mathcal{M}}_*(\rho)) \leq D(\rho\|E_{1*}(\rho)) + D(\rho\|E_{2*}(\rho)) \,, \tag{0.2}$$

where the *coarse-graining maps* [8] $E^{\mathcal{M}}_*, E_{1*}, E_{2*}$ are the Hilbert-Schmidt duals of $E^{\mathcal{M}}, E_1, E_2$. We recover (SSA) from (0.2) by taking $\mathcal{N} \equiv \mathcal{B}(\mathcal{H}_{ABC})$, $\mathcal{N}_1 \equiv \mathcal{B}(\mathcal{H}_{AB})$, $\mathcal{N}_2 \equiv \mathcal{B}(\mathcal{H}_{BC})$ and $\mathcal{M} \equiv \mathcal{B}(\mathcal{H}_B)$.

Conditional expectations arising from the large time limit of a dissipative evolution on subregions of a lattice spin system generally do not satisfy the commuting assumption. In this case, approximations of the (SSA) were found in the classical case for $\mathcal{M} \equiv \mathbb{C}\mathbb{1}_{\mathcal{H}}$ [5] and were recently generalized to the quantum setting in [4, 3, 1, 6]. These inequalities, termed as *approximate tensorization of the relative entropy*, take the following form

$$D(\rho\|\sigma) \leq \frac{1}{1-2c}\left(D(\rho\|E_{1*}(\rho)) + D(\rho\|E_{2*}(\rho))\right) ,$$

where $\sigma := E^{\mathcal{M}}_*(\rho)$ and c is measures the distance from the commuting assumption. Typically, $c = 0$ at infinite temperature, and remains small for conditional expectations onto far apart regions and at high enough temperature.

In this paper, we take one step further and prove a *weak approximate tensorization* for the relative entropy, which amounts to the existence of positive constants $c \geq 1$ and $d \geq 0$ such that

$$D(\rho\|E^{\mathcal{M}}_*(\rho)) \leq c\left(D(\rho\|E_{1*}(\rho)) + D(\rho\|E_{2*}(\rho))\right) + d \,. \tag{AT(c, d)}$$

Here, we estimate both constants c and d in terms of the interactions appearing in the Hamiltonian of the system, more specifically in terms of conditions of clustering of correlations in the setting of quantum lattice spin systems. Our main application of these inequalities is in the context of mixing times of quantum lattice spin systems, as their classical analogues have proven to be a key step in modern proofs of the logarithmic Sobolev inequality for classical lattice spin system. However, we expect these inequalities and their proof techniques to find other applications in quantum information theory.

[1]arXiv: 2001.07981

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