

Contextuality and memory cost of simulation of Majorana fermions

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Contextuality has been reported to be a resource for quantum computation [1], analogous to non-locality which is a known resource for quantum communication and cryptography [2]. In recent work, Karanjai et al. [3] show that the presence of contextuality places a lower bound on the spatial complexity of classically simulating quantum processes. In particular, they bound the memory cost of simulating Clifford operators. The method to derive lower bound works for a general family of classical simulations that can be framed as stochastic evolution in ontological models (hidden variable models): For instance, it generalizes the Gottesman-Knill [7] and Wigner function simulation methods. However, the techniques in [3] are limited to quantum processes that of closed sub-theories, i.e., where products of measurable quantities are measurable, which are a very limited class of quantum circuits. Consequently, it excludes processes where measurements are loyal.

In this work, we generalize this connection to non-closed circuit families with a broader concept of contextuality, namely, event-based contextuality [4]. We show that the presence of “event-based” contextuality places new lower bounds on the memory cost for simulating restricted classes of quantum computation. We apply this result to the simulation of the restricted model of quantum computation based on the braiding of Ising anyons known as topological quantum computation (TQC) model [5]. This model is the first known scheme of magic states distillation [6], a leading paradigm in fault-tolerant quantum computing. It is also of fundamental interest in the study of quantum resources that power quantum computation, as it lies at the intersection of two classically simulable sub-theories: FLO and Clifford circuits. For the TQC model, we prove that the lower bound in the memory required in a simulation is $\Omega(n \log_2 n)$, where n is the number of fermionic modes. This bound is extended to fermionic linear optics (FLO), a fermionic analogous of bosonic linear optics.

Lower bound in the memory cost

Since our goal is to simulate quantum statistics, the ontological model used for the classical simulation will reproduce the Born rule probabilities of a quantum sub-theory. In the classical simulation the density matrix is represented by a probability distribution $\mu_\rho(\lambda)$ over the state space Λ and the measurements become sub-stochastics maps, $\Gamma_O(\lambda', k|\lambda)$. After a measurement the probability distribution $\mu_\rho(\lambda)$ is updated to $\mu_{\rho'}(\lambda')$ with probability $Pr(O, k|\rho, \lambda) = \sum_{\lambda, \lambda'} \Gamma_O(\lambda', k|\lambda) \mu_\rho(\lambda)$. The internal state $\lambda \in \Lambda$ contains all the information necessary to characterize the statistics of all measurements allowed in the sub-theory. The lower bound in the space complexity is obtained by find-

ing a lower bound in the size of the state space Λ required to simulate the sub-theory.

We can define a *sub-theory* as a set of observables \mathbb{O} that one can measure in a class of experiments. In a fixed sub-theory we consider a set of quantum states $S = \{\rho_i\}$ and the set of all observables \mathbb{O}_S that have at least one eigenstate in S . We show that if \mathbb{O}_S is event-contextual, then $\bigcap_s \text{supp}(\mu_{\rho_i}) = \emptyset$ for any simulation of this sub-theory. States that are not single-shot distinguishable [3] must be classically represented by probability distributions that have intersecting supports, i.e., these states must share at least one internal state λ . This limits the size of the state space. By considering the sub-theory defined by the set of quantum states S_{total} , we prove that the lower bound for the size of the state space of this sub-theory is $|\Lambda| \geq S_{total}/m$ where m is the cardinality of the largest set $S \subset S_{total}$ that has a corresponding set of observables \mathbb{O}_S that are non-contextual. Therefore, the memory cost of simulating this sub-theory is lower bounded by $\log_d(|\Lambda|) \geq \log_d(S_{total}) - \log_d(m)$.

Application to the TQC and FLO model

In the TQC model framework [5], the initial state can be mapped to other states by the use of braid gates and measurable observables. The states allowed in the sub-theory can be stabilized by pairs of Majorana operators, e.g., $C = (-im_1m_2, -im_3m_4, \dots, -im_{2n-1}m_{2n})$. Majorana operators have obey commutation rules $m_i m_j + m_j m_i = 2\delta_{ij}\mathbb{I}$, $m_i^\dagger = m_i$, for any i, j .

We show that the lower bound in the memory cost for the simulation of Majorana fermions scales in $\Omega(n \log_2 n)$ in the number of fermionic modes. We also show that the scaling is optimal using the classical simulation methods in [7].

Quantum computation with fermionic linear optics can be seen as a generalization of the TQC, where the unitaries, called FLO gates, are not restricted to the $\pi/4$ angle [8]. We prove that the bound computed for TQC can be extended for FLO. Consequently, the lower bound in the spacial complexity of simulating quantum computation with FLO is $n \log_2 n$.

Thus, our work establishes a connection between contextuality and memory cost of classically simulating quantum circuits. We do so for the most minimal scheme of magic state distillation in a physically motivated setting. We develop new techniques to derive lower bounds in the memory cost of simulating physical sub-theories and apply them to fermions for the first time.

References

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A Introduction

Since the manuscript of the submitted work is not published yet, we provide here a brief comment on the method and the results.

B Ontological model

We start by introducing the notion of sub-theory and defining the requirements of a classical algorithm that simulates the statistics of a measurement outcome.

We can define a *sub-theory* as a set of observables \mathbb{O} that one can measure in a class of experiments. A sub-theory can be characterized by its statistics, i.e., the probability of obtaining specific outcomes when measuring an observable. We consider a state-space Λ , where $\lambda \in \Lambda$ is an internal state (or ontic state), which encodes all the information needed to determine the statistics of the sub-theory. The cardinality (size) of the state-space, $|\Lambda|$, determine the spatial complexity of the simulation, i.e., number of classical bits of memory required for the simulation. The density matrix is represented, in the classical simulation, by a probability distribution $\mu_\rho(\lambda)$ over the state space. A set of von Neumann instruments (CP maps), $\mathcal{M} = \{\mathbb{P}_k^O\}_k$, become a set of classical instruments (sub-stochastics maps), $\mathcal{M}^{cl}\Gamma_O(\lambda', k|\lambda)$. After a measurement the probability distribution $\mu_\rho(\lambda)$ is updated to

$$\mu_{\rho'_k}(\lambda') = \frac{\sum_{\lambda \in \Lambda} \Gamma_O(\lambda', k|\lambda) \mu_\rho(\lambda)}{\sum_{\lambda, \lambda'} \Gamma_O(\lambda', k|\lambda) \mu_\rho(\lambda)}, \quad (1)$$

with probability

$$\Pr(O, k|\rho, \lambda) = \sum_{\lambda, \lambda'} \Gamma_O(\lambda', k|\lambda) \mu_\rho(\lambda). \quad (2)$$

For the sub-theories considered in this work, the classical simulation can be summarized by these three objects $(\Lambda, \{\mu_\rho\}, \{\Gamma_O\})$, the state space, the set of probability distributions over the state space and the stochastic maps. These tools allow us to classically simulate sequences of measurements in the sub-theory.

C Lower bound in the memory cost

In this section, we show the connection between the size of the internal state space and contextuality. We start by presenting the definition of single-shot distinguishability and partitioning measurement, and its relation with event-based contextuality.

If it is possible, within the sub-theory, to perform a measurement that perfectly distinguishes between two different states ρ and ρ' , then we say these states are single-shot distinguishable (SSD). Accordingly, in the classical simulation of this sub-theory, the support of the probability distributions of SSD states is disjoint

$$(\text{SSD}) \quad \implies \quad \text{supp}(\mu_\rho) \cap \text{supp}(\mu_{\rho'}) = \emptyset. \quad (3)$$

In other words, no internal state can be in the support of two states that are SSD.

The SSD condition already ensures us a lower bound on the cardinality $|\Lambda|$ equal to the number of SSD states in the sub-theory. However, some non-SSD states must also have non-intersecting support of distributions.

Consider a set of quantum states $S = \{\rho_i\}$ and the projective measurements of an observable O with outcome k that maps the states in S to the set of post-measurement states $S_{O,k} = \{\sigma_{k,i}\}$. The observable O is a partitioning measurement for S if the set of post-measurement states contains at least one pair of orthogonal quantum states for every outcome k . Therefore,

$$\bigcap_i \text{supp}(\mu_{\sigma_{k,i}}) = \emptyset \quad \forall k. \quad (4)$$

Theorem 1. *If, for a set of quantum states $S = \{\rho_i\}$, exists a partitioning measurement O within the sub-theory, then $\bigcap_i \text{supp}(\mu_{\rho_i}) = \emptyset$ for any simulation of this sub-theory.*

The proof is done by showing that if in the classical simulation, all states in the set S share at least one internal, then the set \mathbb{O}_S is non-contextual.

The existence of a partitioning measurement for a set of state S implies that there is no internal state the is shared by all states. Consequently, this gives us another tool to find lower bound for the size of the state space.

To define *event-contextuality* we first need to introduce the concept of *event*. Consider a commuting o set of observables $\mathbf{A} = (A_1, A_2, \dots, A_k)$, known as context, and a corresponding set of eigenvalues $\mathbf{a} = (a_1, a_2, \dots, a_k)$. Here we represent an *event* as a sequence of projective measurements of a commuting set of observables \mathbf{A} with outcomes

\mathbf{a} , $\mathbb{P}_{\mathbf{a}}^{\mathbf{A}} = \mathbb{P}_{a_1}^{A_1} \mathbb{P}_{a_2}^{A_2} \dots \mathbb{P}_{a_k}^{A_k}$. Let λ be the function that assigns values $(1, 0)$ for events depending on whether they happen or not

$$\lambda(\mathbb{P}_{\mathbf{a}}^{\mathbf{A}}) = \begin{cases} 1, & \text{if } \mathbb{P}_{\mathbf{a}}^{\mathbf{A}} \rho \mathbb{P}_{\mathbf{a}}^{\mathbf{A}} \neq 0 \\ 0, & \text{if } \mathbb{P}_{\mathbf{a}}^{\mathbf{A}} \rho \mathbb{P}_{\mathbf{a}}^{\mathbf{A}} = 0, \end{cases} \quad (5)$$

for a quantum state ρ . The set of observables \mathbf{A} is said to be non-contextual if the value assignment $\lambda(\mathbb{P}_{\mathbf{a}}^{\mathbf{A}})$ is the same even whether is measured with another commuting context or not, e.g.

$$\sum_{\mathbf{b}} \lambda(\mathbb{P}_{\mathbf{a}, \mathbf{b}}^{\mathbf{A}, \mathbf{B}}) = \lambda(\mathbb{P}_{\mathbf{a}}^{\mathbf{A}}). \quad (6)$$

For certain sets of observables it can be shown that a the value assign cannot be independent to the context that is jointly measured. This is a proof of *contextuality*. We say that the set of commuting observables \mathbf{A} exhibits state-independent contextuality if for every state ρ is contextual.

Theorem 2. *Consider a set of quantum states $S = \{\rho_i\}$. Let \mathbb{O}_S be the set of all observable operators that have at least one eigenstate in S . If \mathbb{O}_S is event-contextual, then S allows for partitioning measurement.*

Corollary 3. *If \mathbb{O}_S is event-contextual, then the intersection of the $\bigcap_s \text{supp}(\mu_{\rho_i}) = \emptyset$ for any simulation of this sub-theory.*

The theorems above give us all the tools needed to compute the lower bound in the spatial complexity of simulating classically quantum processes.

Theorem 4. *Consider a sub-theory defined by S_{total} . The lower bound on the size of the state space of the classical simulation of this sub-theory is*

$$|\Lambda| \geq \frac{S_{total}}{m}, \quad (7)$$

where m is the cardinality of the largest set $S \subset S_{total}$ such that the corresponding set of observables, \mathbb{O}_S , are non-contextual, i.e.,

$$m = \max \{ |S| : \mathbb{O}_S \text{ is non-contextual} \}. \quad (8)$$

D Application

We apply our result to the simulation of TQC and FLO.

Bravyi introduced TQC with *Ising anyons* that uses magic states distillation to bring universality to the computation [5]. In this framework, the initial state, $|\mathbf{0}\rangle = |0\rangle^{\otimes n}$, can be mapped to all the other states by the use of braid gates, $B_{ij} = \exp(-\frac{\pi}{4} m_i m_j)$, and measurable observables, $F_{ij} = -i m_i m_j$, where m_i are Majorana operators. Majorana operators have the property that $m_i m_j + m_j m_i = 2\delta_{ij} \mathbb{I}$ for any i, j . The states allowed in the sub-theory can be stabilized by pairs of Majorana operators, e.g.,

$$C = (-i m_1 m_2, -i m_3 m_4, \dots, -i m_{2n-1} m_{2n}). \quad (9)$$

Applying any sequence of braid gates B_{ij} to the state above is equivalent to updating the stabilizer group

$$C' = (-i m_{\pi(1)} m_{\pi(2)}, \dots, -i m_{\pi(2n-1)} m_{\pi(2n)}), \quad (10)$$

where π is a permutation of the numbers $\{1, 2, \dots, 2n\}$. Consequently, the number of states allowed in the sub-theory is

$$|S_{total}| = \frac{(2n)!}{n!}. \quad (11)$$

The first ingredient to compute the size of the state space is the total number of states in the sub-theory. The next step is to find the cardinality of the largest set of states that correspond to a non-contextual set of observables. We show the cardinality of this set is upper bounded by a set of $n(n-1) + 1$ pairwise non-orthogonal states.

Lemma 5. *Any set S of non-orthogonal states with cardinality $|S| \geq n(n-1) + 1$ have a corresponding set \mathbb{O}_S that contains all measurable observables in the Majorana sub-theory.*

In a companion work we show together with collaborators that Majorana fermions sub-theory exhibits contextuality for $n \geq 3$. Consequently, a set of $n(n-1) + 1$ pairwise non-orthogonal states will correspond to a set of observables that is contextual. Consequently, $m < n(n-1) + 1$ and

$$|\Lambda| \geq \frac{|S_{total}|}{m} \geq \frac{|S_{total}|}{n(n-1) + 1}. \quad (12)$$

Theorem 6. *The number of internal states required by a classical simulation of the Majorana fermion sub-theory is*

$$|\Lambda| \geq \frac{2^n}{n(n-1) + 1} \prod_{k=1}^n (2k-1). \quad (13)$$

The memory required to store these internal states is lower bounded in $\Omega(n \log_2 n)$.

TQC with Ising anyons can be seen as a special case of the quantum computation with fermionic linear optics (FLO). Hence, this result can be extended to the classical simulation of FLO by discretizing the number of states allowed in the FLO sub-theory via ϵ -approximation. The number of ϵ -approximate states is proportional to $(2n)!/n!$, consequently all the results described above can be applied for this sub-theory, given rising the same lower bound in the memory cost of classically simulating the sub-theory.