Quantum advantage from energy measurements of many-body quantum systems

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Motivation: Impressive recent developments in experimental quantum physics are enabling the manipulation of quantum systems of larger and larger sizes. The exploration of quantum systems at this scale can lead to new insights into open questions in many-body quantum physics, which are out of the reach of currently known simulation methods using classical computers. Indeed, several experimental demonstrations of large-scale quantum simulators that outperform certain classical simulations methods have already been reported [2, 3]. Unfortunately, the evidence for quantum advantage in these experiments is based solely on numerical benchmarks against available classical algorithms such as, e.g., DMRG [2]. Hence, this does not exclude the possibility that a *future classical algorithm* is found to perform as efficiently as a given quantum simulator.

For this reason, it is of utmost importance to put statements about quantum advantage on rigorous mathematical ground. This has been the subject of several recent works which demonstrate, based on strong complexity-theoretic evidence, that there are certain tasks that can be performed efficiently by quantum devices for which an efficient classical algorithm cannot exist. These are based on sampling problems that exhibit certain robustness against noise and are tailored to near-term hardware. Examples include boson sampling [4], IQP sampling [5], sampling from random quantum circuits [6, 7] and quantum simulations of constant-time Hamiltonian evolutions [8, 9]. The key strength of these results is that the existence of a quantum advantage is provable assuming plausible complexity theoretic conjectures, such as the non-collapse of the Polynomial Hierarchy [4, 5]. The prospect of demonstrating quantum speed-ups in a reliable way has initiated a new field of theoretical and experimental activity coined *quantum computational advantage* (or quantum "supremacy") [10, 11]. All these works, however, mostly rely on "unphysical" sampling problems that were discovered for the sole purpose of demonstrating a quantum advantage, and further connections with more physically-motivated questions are yet to be explored.

The work of Ref. [12] can be seen as a first attempt to establish such a connection, in the context of a molecular spectroscopy problem. Therein, the authors show the problem of sampling from the vibronic spectrum of a molecule could be solved by a "quantum supremacy device" (a boson sampler) but no rigorous hardness proofs for the spectroscopy problem at hand are presented.

The main aim of our work is to provide a rigorous connection between quantum advantage results and physically-motivated problems. Namely, we focus on the problem of sampling measurement outcomes of a physical observable, such as the Hamiltonian of a many-body quantum system. Our main result is to show that it is classically hard to simulate measurements of simple families of local Hamiltonians (e.g., with nearest neighbour interactions on the 2D square lattice) for certain regimes of measurement resolution and even in the presence of certain types of noise, assuming standard complexity theoretic conjectures. On the other hand, we demonstrate that such measurements can be efficiently implemented on a quantum device. This leads to a conceivable quantum advantage

proposals based on measurements of many-body Hamiltonians.

Problem: Following Refs. [13–15], we characterize the quality of a measurement by the *measurement resolution* δ, which sets the smallest measurement unit, and the *measurement confidence* η. For example, an energy measurement of an eigenstate $|\psi_E\rangle$ with energy E is said to have resolution δ and confidence η if it outputs an estimate E' such that

$$\Pr(|E' - E| \leq \delta) \geq \eta. \tag{1}$$

A generalization of Eq. (1) for arbitrary input states defines the target probability distribution we would like to sample from. The finite resolution and measurement confidence result from natural limitations such as a finite measurement time or energy, which are present even in a noiseless measurement device. In addition, to take into account the unavoidable presence of noise in the implementation of a realistic measurement, we introduce the *sampling error* parameter β. This parameter quantifies the deviation in ℓ_1-norm between the target outcome distribution and that of an ideal measurement of resolution δ and confidence η.

All our results concern measurements of *k-local Hamiltonians* acting on n qubits (two level quantum systems) i.e., Hamiltonians of the form $H = \sum_j H_j$ where each term H_j acts on k qubits, for constant $k \in O(1)$. Moreover, we only consider measurements on *product quantum states*, as we would like that the complexity of the problem comes from the Hamiltonian and not the state to be measured.

Terminology: To analyse the hardness of sampling from energy measurements, we consider two regimes of measurement resolution. The first regime, that we define as *standard resolution*, corresponds to the scaling of the parameter δ as $1/\text{poly}(n)$. For general Hamiltonians, this is the best that can be achieved by a quantum device [16] in polynomial time, for example, by estimating the eigenvalues of the time-evolution operator via quantum phase estimation[17]. This can be seen as a coarse-grained energy measurement since, in general, it is not able to distinguish each of the exponentially many eigenvalues. On the other hand, as discussed in [15], in some specific situations one can exploit knowledge of the Hamiltonian to efficiently achieve what we refer to as a *super-resolution measurement*, i.e. an exponentially small measurement resolution $\delta = 1/\exp(n)$.

Furthermore, regarding the sampling error parameter β, we consider two regimes. We define the measurement as *"approximate"* [4, 5] when the desired sampling error β is required to be a small constant independent of the system size n. Moreover, we define the *"near-exact"* sampling regime if the sampling error β is required to be inverse-exponential in the input size.

Results: Our results on classical hardness of simulating energy measurements concern the previously defined regimes of resolution and sampling errors as summarized in Table I and in more detail below. For the sake of clarity we omit the confidence parameter η, which can be taken to be $\eta = 1 - O(\beta)$. Our main contributions are the following:

(i) We provide quantum advantage protocols for *approximate super-resolution* energy measurements. Specifically, we consider Hamiltonians with nearest-neighbor interactions on 2D lattices that can be efficiently diagonalized on a quantum computer. For the latter, we show, first, that approximate super-resolution measurements can be implemented by building an approximate sampler from the diagonalizing quantum circuit (Theorem 1 of Ref. [1]). At the same time, we prove that these measurements are hard to simulate classically assuming plausible complexity-theoretic conjectures (Corollary 1 of Ref. [1]). This leads to a verifiable quantum advantage result based on energy measurements that could be feasibly implemented in available quantum simulators. These

	Super-resolution $\delta = 1/\exp(n)$	Standard-resolution $\delta = 1/\text{poly}(n)$
Near-exact samp. $\beta = 1/2^{\text{poly}(n)}$	(i) PH-collapse	(ii) PH-collapse
Approx. samp. $\beta = \text{const.}$	(i) PH-collapse*	(iii) BPP=BQP

Table I: Our results on classical hardness for the energy measurement problem, summarized in points (i)-(iii). For the different regimes of resolution δ and sampling error β, we show the complexity theoretic implications of the existence of an efficient classical algorithm for sampling outcomes of energy measurements, corresponding to the local Hamiltonians we construct. The cells in grey correspond to problems that tolerate a certain amount of noise and can possibly be solved by a realistic (noisy) quantum computer. In particular, in Ref. [1], we describe an efficient quantum protocol for approximate super-resolution energy measurements, which could be used to demonstrate a quantum advantage. This result, marked by "*", requires plausible complexity theoretic assumptions other than the collapse of the Polynomial Hierarchy (PH).

results exploit a connection between quantum advantage proposals based on simulating constant-time Hamiltonian dynamics [8, 9] and energy measurement problems.

(ii) Super-resolution measurement procedures for *arbitrary* Hamiltonians are unlikely to exist based on complexity theoretic evidence [15, 18]. For this reason, we investigate the hardness of energy measurements with *standard resolution*. Namely, we give complexity-theoretic evidence that classical computers cannot efficiently simulate energy measurements with standard resolution, even for simple translation-invariant nearest-neighbor Hamiltonians on the square lattice (Theorem 3 of Ref. [1]). Analogously to results obtained in Refs. [19–21] for other sampling problems, our hardness result is valid in the *near-exact* sampling regime where $\beta = 0$ or is inverse-exponential. We give two hardness proofs, one being based on the quantum advantage proposal from two-dimensional IQP circuits [9], the other being based on circuit-to-Hamiltonian constructions [22]. This near-exact sampling problem can be solved efficiently by an ideal (noiseless) quantum computer. However, it is unlikely that this is possible in a realistic, even using fault-tolerant techniques.

(iii) Ideally, one would like a physically motivated quantum advantage experiment based on *approximate sampling problems with standard resolution*, which are more resilient to imperfections. However, we argue that, with current techniques, it is not possible to link the classical hardness of this problem to a Polynomial Hierarchy (PH) collapse as in Refs. [4, 5]. As an intermediate step, we provide alternative hardness results inspired by the BQP-hardness of this problem [13, 14]. Using circuit-to-Hamiltonian constructions [22], we show that a hypothetical classical simulator for energy measurement of local Hamiltonians could be used to approximate arbitrary marginals of the output distribution of any poly-sized quantum circuit (Theorem 4 of Ref. [1]). Based on hardness of simulating universal quantum circuits [6, 23–25], these results give evidence that approximately measuring a local Feynman-Kitaev Hamiltonian in the standard resolution regime is classically intractable. As we will discuss in our manuscript, an open challenge in complexity theory would be to tie these hardness results to a Polynomial Hierarchy (PH) complexity-theoretic collapse or other more standard complexity thoeretic conjectures. Such a result could have additional implications for the development of quantum protocols exhibiting physically-motivated quantum advantages.

Conclusion: Overall, we believe our work brings a new perspective to problem of demonstrating quantum advantage and poses interesting new questions related to the complexity of simulating measurements of (random, local) Hamiltonians. Furthermore, we introduce the concept of quantum

diagonalizable Hamiltonians, a class of Hamiltonians that are exponentially fast-forwardable (in the sense of Atia-Aharonov [15]), and highlight the importance of these properties towards constructing new examples of quantum computational advantage that are robust to experimental imperfections. This could further inspire novel quantum algorithms to expand our knowledge of the inner structure of Hamiltonians as well as new demonstrations of quantum advantage for measuring other quantities of interest in quantum many-body physics.

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