Quantum Magic Rectangles: Characterisation and Application to Certified Randomness Expansion

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Abstract. We study a generalisation of the Mermin–Peres magic square game to arbitrary rectangular dimensions. We characterise these in terms of their optimal win probabilities for quantum strategies. We find that for dimensions at least 3×3, quantum strategies can win with certainty; for dimensions $1 \times n$, they do not outperform classical strategies; for dimensions $2 \times n$, we give lower/upper bounds that both outperform the classical strategies. Finally, we apply our findings to certified randomness expansion. We first find the winning probability of games having a distinguished input with deterministic outcome, and then give robustness and rates as in Miller and Shi (2017).

Keywords: Magic square game, nonlocality, certified randomness expansion.

The Mermin–Peres magic square game [2,3] has played an important role in the foundations of quantum theory. It is a simple two-party game that can be won with certainty by quantum parties sharing entanglement, yet it can only be won with smaller probability by classical parties sharing randomness. This property is also known as quantum pseudo-telepathy, and it can be used as an example of strong contextuality.

In this contribution, we introduce a generalisation of this game to rectangular dimensions, characterise its winning probabilities, and apply the results to certified randomness expansion.

1 Magic Rectangles: Definition & Properties

An $m \times n$ rectangle is fixed, along with sequences $\alpha_1, \alpha_2, \dots, \alpha_m$ and $\beta_1, \beta_2, \dots, \beta_n$ from $\{+1, -1\}$ such that: $\alpha_1 \alpha_2 \dots \alpha_m \cdot \beta_1 \beta_2 \dots \beta_n = -1$. This fixes the game. Two players, Alice and Bob, are given an empty row/column of this rectangle which they must fill according to the rules: (i) Each cell must be from $\{+1, -1\}$, (ii) the product of Alice's/Bob's entries should be α_i, β_j. The players win the game if they filled the common cell with the same value.

Properties: We note that a game is fixed by the dimension of the rectangle and the sequences of α's and β's. We prove that the optimal winning probabilities for any set of behaviours (classical, quantum, almost quantum, non-signalling) are: (1) the same for all games of the same dimension, (2) symmetric with respect to row/column exchange, and (3) monotonically increasing with the dimension of the rectangle.

2 Magic Rectangles: Characterisation

The regular magic square game is a 3×3 magic rectangle game and can be won for quantum strategies with certainty. We use this and the above properties to reduce the full characterisation of magic rectangles to that of $1 \times n$ and $2 \times n$ games. We also show that the CHSH game is a 2×2 magic rectangle game. We then obtain optimal winning probabilities for the $1 \times n$ case which coincide with classical, and we lower and upper bound the winning probabilities for $2 \times n$ games, where both bounds are between classical strategies and unity. For the upper bound, we conjecture the almost quantum [4] winning probabilities based on numerical evidence. As a side result, we get that $2 \times n$ games with $n \geq 3$ can be won with certainty using behaviours at level 1 of the NPA hierarchy [5,6], while the quantum and almost quantum sets both give winning probabilities strictly smaller than unity.

3 Application to Certified Randomness

Finally, we use this characterisation to analyse certified randomness expansion from magic rectangle games. Specifically, we show that the winning probability of an $m \times n$ game with a distinguished input can be obtained from the $(m-1) \times (n-1)$ game. This, along with the previous results, allows us to determine the noise tolerance of each of these games, where it turns out that only $2 \times n$ and $3 \times n$ games can be used for certified randomness expansion. We then follow the analysis of [1] to get rates for certified randomness expansion using different magic rectangle games.

For further details see the full paper [7].

References

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