Covariant Quantum Error Correcting Codes via Reference Frames

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Reliable universal quantum computation requires fault-tolerant error correction [1], the ability to correct errors in the implementation before they propagate too far in the computation. Achieving such quantum error correcting codes (QECCs) is a notoriously challenging task, due to fundamental limitations such as quantum no cloning. One of the earliest proposals to achieve fault-tolerance was the idea of encoding logical gates "transversally" [2, 3]. In such proposals, all gates required to achieve universal computations would be realised at the physical level by applying a tensor product of local gates, with each local gate traversing a set of "code blocks". Such schemes are favourable for keeping error propagation low, since if one can correct for local errors in each code block, one can correct for errors due to an erroneous implementation of one of the local gates. Unfortunately, Eastin and Knill proved that this strategy is destined to fail for a universal quantum computer [4]: If local errors in the code blocks are correctable and the code is finite dimensional, then it cannot admit a universal set of transversal gates.

A very recent proposal, [5], circumvents the Eastin-Knill theorem by employing an infinite dimensional code space using reference frames. The resulting codes are covariant, since the transversality of all unitary gates implies that the action of the gate commutes with the encoder. While impractical, it paved the way for another approach to circumvent the Eastin-Knill theorem: using finite dimensional reference frames and decoders which only recover approximately. As long as the recovery errors are small enough, such errors will be admissible. This latter approach was first considered in [6] for codes covariant under $\mathsf{U}(1)$.

In this work, we use finite reference frames to construct $\mathsf{SU}(d)$ covariant codes from arbitrary non-covariant codes. Our approach, whose details can be found in the full paper, is to divide all the resource qudits into two registers: a physical register, on which we construct the original non-covariant code, and a reference frame register. We study the benefits and limitations to this approach. In particular, we show that their error scaling is optimal in two different commonly-studied erasure error models. In the first model, the erasure occurs on only one or a few qudits. This model was commonly adopted in the literature of covariant error correction [5–7] but is quite demanding on the fidelity of the qudits. In the second, more practical model that we consider, every qudit has a probability of being erased. To distinguish between them, we call the first the *weak error model* and the second the *strong error model*.

In the weak error model, we couple the qudits allocated to the reference frame register and construct highly entangled quantum reference frames. With these reference frames, our protocol yields a covariant code with an error rate that matches the Heisenberg limit of quantum metrology. More precisely, with n being the total number of qudits and n_{e} being the (maximum) number of erased qudits, our covariant code has an error (evaluated in terms of the diamond norm) that scales as $(n_{\mathrm{e}}/n)^2$.

In the strong error model, we divide the reference frame register into multiple quantum reference frame states. Our protocol achieves almost $1/n$ scaling as opposed to the $1/n^2$ scaling in the weak error model. This is a fundamental restriction imposed by the error rather than a deficiency of our protocol. In fact, we prove that the $1/n^2$ error scaling in the weak error model and the $1/n$ error scaling in the strong error model are both optimal. The derivation of both limits holds for other constructions of covariant codes that do not necessarily involve reference frames. Our result has possible applications to fault-tolerant quantum computing, reference frame alignment, and the

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