Algebraic complete axiomatisation of ZX-calculus with a normal form via elementary matrix operations

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Abstract

In this paper we give a complete axiomatisation of qubit ZX-calculus via elementary transformations which are basic operations in linear algebra. This formalism has two main advantages. First, all the operations of the phases are algebraic ones without trigonometry functions involved, thus paved the way for generalising complete axiomatisation of qubit ZX-calculus to qudit ZX-calculus and ZX-calculus over commutative semirings. Second, we characterise elementary transformations in terms of ZX diagrams, so a lot of linear algebra stuff can be done purely diagrammatically.

Keywords: Quantum computation, ZX-calculus, normal form, elementary matrices.

The full paper associated with this extended abstract is available at https://arxiv.org/abs/2007.13739

ZX-calculus was introduced by Coecke and Duncan [3] as a graphical language for quantum computing. It has exhibited its power in the application field of quantum circuit optimisation [1, 8, 4]. On the theoretical side, ZX-calculus has been first proved to be complete for overall qubit quantum computing in [9] and incorporated in [5], which means quantum computation done by matrices can purely be done in ZX-calculus. Afterwards, there came a few different complete axiomatisations of ZX-calculus [6, 7, 10, 11]. However, all of these universal axiomatisations rely either on some non-algebraic rule which has some trigonometry functions involved in, or on the completeness of ZW-calculus [5].

In this paper, we give an algebraic complete axiomatisation (i.e. without non-algebraic rules) of ZX-calculus without resort to the completeness of ZW-calculus. Instead, based on the representation of elementary matrices in ZX-calculus, we obtain a normal form for any vectors, then any matrix can be represented by a diagram due to the map-state duality. This normal form leads to a proof of completeness, though full of nontrivial techniques. In addition, we gain an advantage that no scalable techniques are needed, as considered in [2].

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