Noncontextuality Inequalities from Antidistinguishability

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Noncontextuality inequalities are usually derived from the distinguishability properties of quantum states, i.e. their orthogonality. Here, we show that *anti*distinguishability can also be used to derive noncontextuality inequalities. The Yu-Oh 13 ray noncontextuality inequality can be re-derived and generalized as an instance of our antidistinguishability method. For some sets of states, the antidistinguishability method gives tighter bounds on noncontextual models than just considering orthogonality, and the Hadamard states provide an example of this. We also derive noncontextuality inequalities based on mutually unbiased bases and symmetric informationally complete POVMs.

Quantum contextuality has its origins in work of Bell [1], and Kochen and Specker [2], where they proved a no-go theorem ruling out deterministic hidden variable theories in which the value assigned to an observable is independent of how you measure it. In recent years, contextuality has attracted increasing attention for its role in quantum information processing advantages [3–10] and explaining the power of quantum computation [7, 11–19]. For these purposes, it is useful to find new classes of noncontextuality inequalities and to find the tightest possible bounds on them.

Noncontextuality inequalities are usually based on the orthogonality properties of sets of quantum states. A powerful method for deriving bounds on noncontextuality inequalities from the orthogonality graphs of events has been developed by Cabello, Severini and Winter (CSW) [20, 21].

In this talk, we report on work published in [23] showing that the *anti*distinguishability properties [24] [25] of quantum states can also be used to derive noncontextuality inequalities. The idea of antidistinguishability is that if one of the states $|a_1\rangle, \cdots, |a_n\rangle$ is prepared and you do not know which then there exists a measurement that allows you to definitively rule out one of the states. Our method reproduces the inequality used in the Yu-Oh 13 ray proof of contextuality [26], giving more intuition behind its structure and allowing us to propose several generalizations. In some cases, when we apply both the CSW method and our method to the same set of states, we get a much tighter bound on the noncontextuality inequality. An example of this is given for noncontextuality inequalities based on Hadamard states [27–29].

The antidistinguishability inequalities considered here were first introduced as overlap bounds on the reality of the quantum state [30–33] in the wake of the Pusey, Barrett and Rudolph (PBR) theorem [34]. Our main result is to re-derive these inequalities as noncontextuality inequalities. We also re-derive and generalize some other noncontextuality inequalities that have appeared in the literature [26, 35] by showing that they are examples of the antidistinguishability-based construction.

The main definitions and results of our work are as follows.

Definition 1. A *contextuality scenario* $\mathfrak{C}$ is a structure $\mathfrak{C} = (X, \mathcal{M}, \mathcal{N})$ where

- X is a set of *outcomes*.
- $\mathcal{M}$ is a set of subsets of X such that if $M, M' \in \mathcal{M}$ then $M' \not\subset M$. An $M \in \mathcal{M}$ is called a *(measurement) context*.
- $\mathcal{N}$ is a set of subsets of X such that if $M \in \mathcal{M}$ then $M \notin \mathcal{N}$ and if $N, N' \in \mathcal{N}$ then $N' \not\subset N$. An $N \in \mathcal{N}$ is called a *maximal partial (measurement) context*.

The idea of a contextuality scenario is that you have a system on which you can perform several different measurements. X is the set of all possible measurement outcomes. A context $M \in \mathcal{M}$ is the full set of distinct outcomes that can occur in some possible measurement. A maximal partial context $N \in \mathcal{N}$ is a set of outcomes that can occur as the outcome of some possible measurement, but not necessarily the full set.

Definition 2. A *strong pairwise antiset* W in a contextuality scenario $\mathfrak{C} = (X, \mathcal{M}, \mathcal{N})$ is a set of outcomes for which there exists a context $M \in \mathcal{M}$ such that, for every $a, b \in W$ and $c \in M$, the triple $\{a, b, c\}$ is antidistinguishable.

Definition 3. A *weak pairwise antiset* W in a contextuality scenario $\mathfrak{C} = (X, \mathcal{M}, \mathcal{N})$ is a set of outcomes for which there exists another outcome $c \in X$ such that, for every $a, b \in W$, the triple $\{a, b, c\}$ is antidistinguishable.

The outcome c is called a *principal outcome* for the pairwise antiset W.

We are now in a position to state our main result.

Theorem 4. *Let W be a pairwise antiset in a contextuality scenario $\mathfrak{C} = (X, \mathcal{M}, \mathcal{N})$. If W is strong then any noncontextual state ω satisfies*

$$\sum_{a \in W} \omega(a) \leq 1. \tag{1}$$

If W is weak then any noncontextual state ω that also satisfies $\omega(c) = 1$ for a principal outcome c satisfies eq. (1).

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