Canonical forms of two-qubit states under local operations

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Abstract: We provide a comprehensive analysis to obtain algebraically distinct canonical forms of two-qubit density matrices under local operations. We identify two inequivalent canonical forms realized through Lorentz singular value decompositions of the real parametrization of two-qubit density matrix. An elegant geometric visualization of two-qubit states on and within the Bloch ball follows naturally from our approach.

Keywords: Local operations, Canonical forms, Lorentz invariance, Bloch ball

The action $\rho_{AB} \to A \otimes B\, \rho_{AB} A^\dagger \otimes B^\dagger$ (upto a normalization factor) of stochastic local operation and classical communication (SLOCC) on a two-qubit density matrix $\rho_{AB} = \frac{1}{4}\sum_{\mu,\,\nu=0}^{3} \Lambda_{\mu\,\nu}\,(\sigma_\mu \otimes \sigma_\nu)$ manifests as Lorentz transformation on the 4×4 real matrix Λ (defined by $\Lambda_{\mu\,\nu} = \mathrm{Tr}\,[\rho_{AB}\,(\sigma_\mu \otimes \sigma_\nu)]$) in the form $\Lambda \to \tilde{\Lambda} = L_A \Lambda L_B^T$, where $A, B \in$ SL(2,C) and $L_A, L_B \in$ SO(3,1). With the help of suitable Lorentz transformations L_{A_c}, L_{B_c},it is possible to arrive the Lorentz singular value decomposition $\Lambda^c = L_{A_c}\,\Lambda\, L_{B_c}^T$ of the real matrix Λ parametrizing the two-qubit system. Verstraete et.al. [1, 2] had arrived at two different types of Lorentz singular value decomposition for the real matrix Λ under SLOCC, employing highly technical results on matrix decompositions in n dimesnional spaces with indefinite metric [3]. It was pointed out that the canonical forms of Refs.[1, 2] fail to reveal the underlying geometric features in an unambiguous way [4]. In this work we have carried out a complete anlaysis for obtaining the algebraically different canonical forms of two-qubit states based on an entirely different approach – inspired by the techniques developed in classical polarization optics by some of us [5, 6]. Our approach leads to an elegant geometrical representation of two-qubit state on and within the Bloch ball [7].

We construct two real symmetric matrices $\Omega_A = \Lambda\, G\, \Lambda^T$, and $\Omega_B = \Lambda^T\, G \Lambda$ where $G = \mathrm{diag}(1,-1,-1,-1)$ denotes the Minkowski metric. These matrices undergo Lorentz congruent transformations $\Omega_{A,B} \longrightarrow \tilde{\Omega}_{A,B} = L_{A,B}\,\Omega_{A,B}\, L_{A,B}^T$. Following the detailed mathematical analysis carried out in Refs. [5, 6], we recognize that the matrices $G\,\Omega_{A,B}$ [7] are *positive semidefinite* and they have *either time-like or light-like Minkowski eigenvectors associated with their highest eigenvalue.* This results in *two* distinct canonical forms $\Lambda^{\mathrm{I_c}}$ or $\Lambda^{\mathrm{II_c}}$ for the real matrix Λ. The type-I canonical form, corresponding to time-like eigenvector associated with the highest eigenvalue λ_0 of $G\Omega_{A,B}$, is diagonal $\Lambda^{\mathrm{I_c}} = \mathrm{diag}\,(1, \sqrt{\lambda_1/\lambda_0}, \sqrt{\lambda_2/\lambda_0}, \sqrt{\lambda_3/\lambda_0})$ and the associated two-qubit state is in the Bell-diagonal form [7]. On the other hand the type-II canonical form $\Lambda_A^{\mathrm{II_c}}$ (corresponding to light-like eigenvector for the highest eigenvalue of $G\Omega_{A,B}$ has a non-diagonal form given by [7] $\Lambda_A^{\mathrm{II_c}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & r_1 & 0 & 0 \\ 0 & 0 & -r_1 & 0 \\ 1-r_0 & 0 & 0 & r_0 \end{pmatrix}$, where $0 \leq r_1^2 \leq r_0 \leq 1$, $r_\alpha = \frac{\lambda_\mu}{\phi_0}, \mu = 0,1$, $\phi_0 = \left(L^T A\,\Omega_A\, L_A\right)_{00}$, with $\lambda_0 \geq \lambda_1$ denoting doubly degenerate eigenvalues of $G\Omega_A$. We show that under the map $\Lambda^{\mathrm{I_c}}$ the set of all four-vectors $\left\{(1,x_1,x_2,x_3)^T \left| x_1^2+x_2^2+x_3^2 = 1\right.\right\}$ representing the Bloch sphere gets transformed to $\left\{\Lambda^{\mathrm{I_c}}\,(1,x_1,x_2,x_3)^T = (1,y_1,y_2,y_3)^T\right\}$ depicting ellipsoidal surface with semi-axes lengths $\sqrt{\lambda_i/\lambda_0}, i = 1,2,3$, centered at the origin. We also show that the type-II canonical form $\Lambda_A^{\mathrm{II_c}}$ maps the set of all four-vectors $\left\{(1,x_1,x_2,x_3)^T \left| x_1^2+x_2^2+x_3^2 = 1\right.\right\}$ on to a spheroidal surface characterised by the four-vectors $(1,y_1,y_2,y_2)^T$ obeying $\frac{y_1^2+y_2^2}{r_1^2} + \frac{(y_3-(1-r_0))^2}{r_0^2} = 1$. The geometric intuition underlying the canonical forms is displayed in the figure.

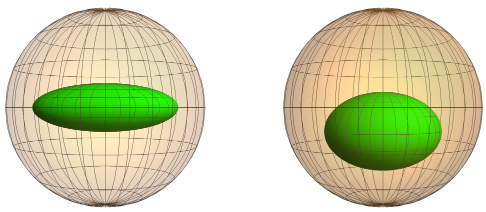

FIG. 1: Ellipsoidal (left) and spherodal (right) surfaces inside the Bloch ball characterizing type-I, II canonical forms $\Lambda_A^{\mathrm{I_c}}$, $\Lambda_A^{\mathrm{II_c}}$ parametrizing the SLOCC inequivalent families of two-qubit density matrices.

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