Finding good quantum codes using the Cartan form [12]

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We present a simple and fast numerical procedure to search for good quantum codes for arbitrary noise processes, using the worst-case fidelity as the figure of merit. We reduce the complexity of the problem by fixing the form of the recovery to be a near-optimal recovery map, adapted to the noise in question. For qubit codes, we obtain a simple form for the objective function, which makes the optimization tractable. We parameterize our search space of encoding unitaries using the Cartan decomposition which allows us to search over the nonlocal parts of the encoded space, leading to families of channel-adapted codes.

Keywords: Channel-adapted quantum codes, numerical optimization, Cartan decomposition.

Motivation: We are today in the so-called "NISQ" era [1], with quantum devices that have small number of noisy qubits, that are not amenable to implement regular quantum error correction (QEC) and fault tolerance schemes. While much of the existing work on QEC revolves around using standard codes [3] that can correct for *arbitrary* errors on individual qubits, this approach may be resourceful, requiring atleast five physical qubits to protect one qubit. However, when we have prior knowledge of the noise afflicting the system, codes adapted to the noise (channel-adapted codes) in question are known to be more effective [5, 6, 8]. In our work, we provide a prescription to search for shorter, channel-adapted codes, tailored to deal with specific noise processes.

Finding optimal qubit codes: We focus on finding channel-adapted qubit codes that minimize the worst-case fidelity for the storage of a single qubit of information. For a given pair of encoding $\mathcal{W}$ and recovery R, for a noise process $\mathcal{E}$, *worst-case* fidelity is obtained as

$$F^2_{\min}(\mathcal{W},\mathcal{R};\mathcal{E}) \equiv \min_{|\psi\rangle\in\mathcal{H}_0} F^2(|\psi\rangle, \mathcal{W}^{-1}\circ\mathcal{R}\circ\mathcal{E}\circ\mathcal{W}). \quad (1)$$

where we minimize the fidelity $F^2(.,.)$ over the single qubit state space $\mathcal{H}_0$ that we encode in, and the function F^2 is defined as $F^2(|\psi\rangle,\mathcal{M}) = \langle\psi|\,\mathcal{M}(|\psi\rangle\langle\psi|)\,|\psi\rangle$. In order to obtain the best performing codes one should further optimize the fidelity in Eq. 1 over encodings $\mathcal{W}$ and recovery $\mathcal{R}$, thus leading to a triple optimization. By assuming a specific form for the recovery as the near-optimal Petz map [7] $\mathcal{R}_P$, the optimal encoding for a given noise process $\mathcal{E}$ is obtained as a solution to the optimization of the fidelity loss function [8] as given below,

$$\eta_{\rm op} = \underset{\mathcal{W}}{\rm argmin}\ \frac{1}{2}[1 - t_{\min}(\mathcal{W})]. \quad (2)$$

Here $t_{\min}(\mathcal{W})$ is the smallest eigenvalue of a 3×3 matrix for a given encoding $\mathcal{W}$, which is calculable numerically.

Role of the Cartan decomposition: We make a well motivated assumption about the noise that the full n-qubit noise can be expressed as tensor products of n single qubit channels [11], giving the noise a local character. One then expects to find good code spaces in the regions of state space with non-local structure. This motivates our use of the Cartan decomposition [9] for the encoding unitaries, which breaks down a given unitary as tensor products of alternating single qubit (local) and multi-qubit (non-local) unitaries [10]. This allows us to explicitly search over the nonlocal pieces of the Cartan decomposition, thereby reducing the parameters of the search space, and leading to families of good quantum codes.

Example: We demonstrate the usefulness of our search procedure for the case of the amplitude damping channel in the Fig. 1 below. The figure illustrates how the 4-qubit codes obtained using our procedure outperform the standard 5-qubit code over the entire range of noise parameter γ. Furthermore, using the Cartan form, we also obtain simple encoding circuits for the optimal codes obtained using our search [12].

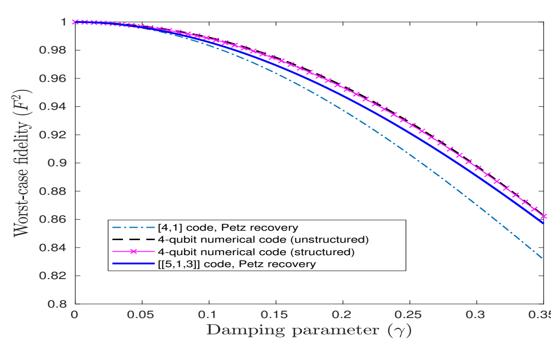

FIG. 1. Codes for the amplitude damping channel.

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