# Rates of multi-partite entanglement transformations

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**Abstract.** The rates at which bi-partite entangled states can be asymptotically transformed are fully determined. In the multi-partite setting, a similar question of the optimally achievable rates of transforming one pure state into another is notoriously open. In this work, we report substantial progress by deriving simple upper and lower bounds on the rates that can be achieved in asymptotic multi-partite entanglement transformations. These bounds are based on ideas of entanglement combing and state merging. We identify cases where the bounds coincide and hence provide the exact rates. This result provides further scope for quantum internet applications beyond points-to-point.

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Entanglement is the feature of quantum mechanics that renders it distinctly different from a classical theory [1]. It is at the heart of quantum information science and technology as a resource. A resource which needs to be used in different form. As a consequence, questions were asked how one form of entanglement could be transformed into another. It was one of the early main results of the field of quantum information theory to show that all pure bi-partite states could be asymptotically reversibly transformed to another bi-partite state with local operations and classical communications (LOCC) at a rate that is determined by a single number [2]. However, the situation in the multi-partite setting is significantly more intricate [3]. The rates that can be achieved when aiming at asymptotically transforming one multi-partite state into another with LOCC are far from clear.

We determine, in this work, simple lower and upper bound for the asymptotic rate of conversion between multi-partite entangled states. The setup is the following: let N be an integer, we consider N+1 parties we divide into one Alice and N Bobs. They share together several copies of a pure state $\psi^{A,B1,...,BN}$, each of the party can act on its part of the state and can communicate classicaly with the others. The aim is the asymptotic conversion of the (N+1)-partite pure state $\psi^{A,B1,...,BN}$ into another pure state $\sigma^{A,B1,...,BN}$, with a rate of conversion as high as possible. Noting $R(\psi^{A,B1,...,BN} \rightarrow \sigma^{A,B1,...,BN})$ as the optimal rate of asymptotic conversion, we prove that:

$$\min_X \frac{S(\psi^X)}{S(\sigma^X)} \geq R(\psi^{AB_1 \ldots B_N} \rightarrow \sigma^{AB_1 \ldots B_N}) \geq \min_X \left\{ \frac{S(\psi^{AX})}{\sum_{B_i \notin X} S(\sigma^{B_i})} \right\}$$

where X denotes a subset of all Bobs. The already known upper bound follows from the fact that any multipartite LOCC protocol is also bipartite with respect to any of the bipartitions and, as previously mentionned, bi-partite asymptotic conversion is fully known. Our main result is the finding of the lower bound, which is optimized by taking the maximum over all possible choices of Alice. It is built upon the machinery of entanglement combing, which was introduced and studied for general N-partite scenarios in [4] and quantum state merging [5]. Entanglement combing is a class of protocol converting multipartite states into a product of bi-partite states, which can be used, along with Schumacher compression, as means to distribute the copies of $\sigma^{A,B1,...,BN}$. Judicious uses of Von Neumann entropy strong subbaditivity and time-sharing allow us to achieve a protocol which reaches the rate of conversion given by the lower bound.

We show the lower bound can be tight with the upper one in practical cases, thus giving the optimal asymptotic rate of conversion. We choose to consider the problem of tri-partite GHZ states distillation. We consider, as an example, the family of states given by $|\psi\rangle^{ABC}=\cos\alpha|0,0,0\rangle+\sin\alpha\sin\beta|0,1,1\rangle+\sin\alpha\cos\beta|1,0,1\rangle$, with $\alpha$ as a real parameter and $\beta$ as 1/2. Computing both the lower bound and the difference between upper and lower bound, we show that the bounds are tight for a large parameter range of $\alpha$, implying that our bound gives the exact conversion rate in these cases. To the best of our knowledge, this outperforms any previously known bounds, such as the long-standing one of Smolin and al. [6] as they consider only one-way broadcasting protocols while ours is not limited to a particular class of LOCC.

It should be clear that the results established here readily allow to assess how resources for multi-partite protocols can be prepared from multi-partite states given in some form. There is still a lot of unknown concerning the manipulation of multi-partite resources. This state of affairs is unfortunate, and even more so since multi-partite resources are at the heart of several quantum applications such as quantum secret sharing [7], quantum voting [8] or distributed quantum computing [9] and implementation of those applications in global quantum networks -- the “quantum internet” [10] -- may become an experimental reality in the not too far future. However, creating entanglement is undeniably costly in quantum networks, prompting the need that previously distributed states are reused as much as possible by converting them into desired resources rather than using more precious entanglement, in schemes involving more than one copy at a time. We hope that our established bounds provide meaningful guidance as to how to manage and recycle resources for quantum networks.

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