Heat exchange and fluctuation in Gaussian thermal states in the quantum realm

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Abstract. The celebrated exchange fluctuation theorem proposed by Jarzynski and Wózcik, (Phys Rev. Lett. **92**: 230602, 2004) for heat exchange between two systems in thermal equilibrium at different temperatures is explored here for quantum Gaussian states in thermal equilibrium. We employ Wigner distribution function formalism for quantum states, which exhibits close resemblance with the classcial phase-space trajectory description, to arrive at this fluctuation theorem. We show that the quantum Jarzyinski- Wózcik theorem agrees with the corresponding classical result in the high temperature limit.
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The fluctuation-exchange relations can be considered as generalizations of second law of thermodynamics for small systems and they connect the probabilities of appearance of physical quantitites such as work, heat, number of particles, in a given set up, to those obtainable in a time-reversed formulation. For instance, the Jarzynski-Wózcik exchange fluctuation theorem (XFT) [1] given by $\frac{p_\tau(Q)}{p_\tau(-Q)} = e^{\triangle\beta\,Q}$, $\triangle\beta = (k\,T_A)^{-1} - (k\,T_B)^{-1}$, quantifies the ratio of probability $p_\tau(Q)$ of heat exchange during interaction of system A (in equilibrium at temperature T_A) and system B (in thermal equilibrium at temperature T_B) for a fixed time duration τ, to its time-reversed counterpart $p_\tau(-Q)$. Here k denotes Boltzmann's constant and Q is the amount of heat exchanged. In the quantum regime, Jarzynski and Wózcik considered systems with discrete energy levels to arrive at an analogous quantum XFT. In this work we retain the flavour of phase-space approach in the quantum scenario, by confining ourselves to continuous variable Gaussian thermal states which are characterized by positive Wigner phase space distributions. We follow analogous steps as that of the original work [1] to arrive at the heat exchange-fuctuation theorem in the quantum realm for two Gaussian states in thermal equilibrium at temperatures T_A, T_B.

Wigner function of ensembles of quantum oscillators A and B prepared in thermal equilibrium at temperatures T_A, T_B respectively is given by

$$W(\zeta) = (4\pi^2\,\nu_{T_A}\nu_{T_B})^{-1}\,\mathrm{Exp}\left(-\frac{H(\zeta_A)}{\hbar\omega_A\,\nu_{T_A}} - \frac{H(\zeta_B)}{\hbar\omega_B\nu_{T_B}}\right), \zeta = (\zeta_A;\ \zeta_B) = (q_A, p_A;\ q_B, p_B)$$

Here $H(\zeta_i) = \frac{P_i^2}{2m_i} + \frac{1}{2}\,m_i\omega_i^2 Q_i^2$, $Q_i = \sqrt{(\hbar/m_i\omega_i)}\,q_i$, $P_i = \sqrt{m_i\omega_i\hbar}\,p_i, i = A, B$ denotes classical Hamiltonian function of oscillators and $\nu_{T_i} = \cot(\hbar\omega_i/k\,T_i), i = A, B$. The systems are kept in contact for a fixed time duration τ where the interaction Hamiltonian is quadratic in phase space operators, so that the canonical commutation relations are preserved. This results in a real linear symplectic transformation $\zeta \to \zeta_\tau = S_\tau\,\zeta$ of the phase space column ζ. Dynamical evolution under time-reversed trajectory $\zeta_0{}^* = \bar{\zeta}_\tau$ to $\zeta_\tau{}^* = \bar{\zeta}_0$ leads to the ratio of Wigner function $\frac{W(\zeta_0)}{W(\bar{\zeta}_0)} = \prod_i e^{\triangle E_i\,Q}$, $\triangle E_i = H(\bar{\zeta}_{i0}) - H(\zeta_{i0})$, i=A,B. One obtains [1, 2] $\triangle E_A \approx -\triangle E_B = Q(\zeta_0)$ as amount of heat exchanged during evolution when interaction energy can be ignored. Consequently, it may be seen that the ratio of Wigner functions is given by $(W(\zeta_0)/W(\bar{\zeta}_0)) = e^{\triangle\beta_\omega\,Q(\zeta_0)}, \triangle\beta_\omega = \left(\hbar\omega_A\,\nu_{T_A}\right)^{-1} - \left(\hbar\omega_B\,\nu_{T_B}\right)^{-1}$. Expressing $p_\tau(Q) = \int d\zeta_0\,W(\zeta_0)\,\delta(Q - Q(\zeta_0))$ and simplifying using time-reversal properties, one obtains the Jarzynski- Wózcik relation

$$p_\tau(Q)/p_\tau(-Q) = e^{\triangle\beta_\omega\,Q},$$

which is structurally similar to the classical XFT. Classical XFT is realized when $\triangle\beta_\omega \to \triangle\beta = (k\,T_A)^{-1} - (k\,T_B)^{-1}$ in the high temperature limit $(\hbar\omega_i/k\,T_i) \to 0$. Deviation of XFT from its classical counterpart can be attributed to the fact that classical equipartition theorem no longer holds in the quantum regime [3].

References

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